

# ON $H^\infty$ OPTIMAL SIGNAL RECONSTRUCTION IN NOISY FILTER BANKS

H. Vikalo, B. Hassibi, and T. Kailath

Information Systems Laboratory  
Stanford University, CA 94305

## ABSTRACT

We study the design of synthesis filters in noisy filter bank systems using an  $H^\infty$  point of view. For unitary analysis polyphase matrices we obtain an explicit expression for the minimum achievable disturbance attenuation. Numerical examples and comparisons with existing methods are also included.

## 1. INTRODUCTION

Multirate filter banks systems have been a subject of extensive studies (see [1]-[3] and the references therein) and are widely used in many application areas (such as speech and image compression, joint source channel coding, adaptive systems, and others). The design of perfect reconstruction filter banks, capable of exactly replicating the input signal, has received particularly high attention. In most of the research, the subbands of the filter bank system are assumed noise free. Figure 1 illustrates such a filter bank with two subband channels. The analysis filters  $H_0(z)$  and  $H_1(z)$  decompose the input signal into subband components, which are then decimated by a factor of 2. The signal is reconstructed by upsampling by a factor of 2 followed by filtering with synthesis filters  $F_0(z)$  and  $F_1(z)$ . The decimated

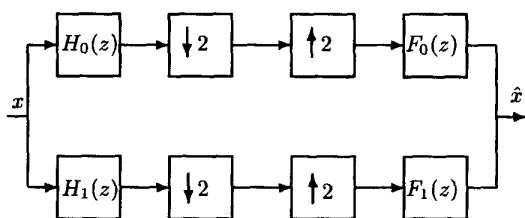


Figure 1: Two-channel filter bank

signals in the subbands may be, for example, encoded and

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transmitted (as in speech comparison applications), or be coded for storage, at which point the signal may be compressed and some information lost. The perfect reconstruction approach studied in the literature, assumes no loss of information in the subbands. However, signal quantization and noise corruption in the subbands, as well as computational roundoff, are always present in practical filter banks systems ([4],[5]).

In order to deal with noise-corrupted filter bank systems, multirate Kalman synthesis filtering has been recently proposed ([7]). In ([8]), methods for optimal signal reconstruction in noisy nonuniform filter banks, using Kalman and  $H^\infty$  filters, have also been proposed for cases when the input signal model is inaccessible. The Kalman filtering approaches require a priori knowledge of the (first and second-order) noise statistics. Therefore in applications involving compression, quantization, etc., where the noise statistics are not readily known, the performance of the synthesis filters may be suspect.

$H^\infty$  estimation, on the other hand, requires no statistical assumptions, performs a worst-case design, and is therefore robust with respect to noise uncertainty. Moreover, the  $H^\infty$  approach allows one to explicitly introduce finite delay into the synthesis filter design. [Conventional IIR perfect reconstruction filters are often non-causal, which requires either infinite delay or some form of truncation.] The  $H^\infty$  optimization approach had been proposed earlier in [6] but, unlike the current paper, considered only the noise-free subband case.

To begin our study, we will use a polyphase representation of the filter bank shown in Figure 1. Performing a type-1 polyphase decomposition of the analysis filters ([1]), one can define the polyphase analysis matrix  $H(z)$  as

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = H(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, \quad (1)$$

while the polyphase synthesis matrix is found by performing a type-2 polyphase decomposition of the synthesis filters,

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} F(z^2). \quad (2)$$

Since we are interested in estimating  $x_{i-d}$ , the delayed version of the input signal ( $d > 0$ ), performing blocking of the

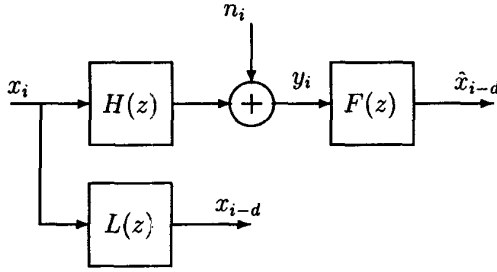


Figure 2: Polyphase equivalent to 2-band noisy filter bank

input and the output of the filter bank leads to the equivalent system in Figure 2, where the delay transfer matrix is of the form ([6])

$$L(z) = \begin{cases} z^{-d} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } m = 2d + 1, \\ z^{-d} \begin{bmatrix} 0 & z \\ 1 & 0 \end{bmatrix} & \text{if } m = 2d. \end{cases}$$

The system in Figure 2 is the standard model for a general estimation problem, where the goal is to design the causal linear time-invariant estimator  $F(z)$  to estimate the input sequence  $\{x_{i-d}\}$  from the observations  $\{y_i\}$ .

We should also mention that extensive studies of orthogonal filter banks (often in regard to perfect reconstruction solutions) can be found in the literature (see, e.g., [1]). A common choice for the analysis filters are those leading to paraunitary (or scaled paraunitary) polyphase analysis matrices. Therefore in the next section we shall consider the design and performance of such filter banks from the  $H^\infty$  estimation point of view.

## 2. $H^\infty$ APPROACH

With the adopted polyphase representation of the filter bank, the design of the synthesis polyphase matrix can be regarded as a special case of the estimation problem formulation, where the observations  $\{y_i\}$  are the noise corrupted subbands signals. The induced transfer matrix mapping the unknown disturbances  $x_i$  and  $\sigma^{-1}n_i$  to the estimation errors is

$$T_F(z) = [L(z) - F(z)H(z) \quad -\sigma F(z)], \quad (3)$$

where  $\sigma^2$  represent the intensity of the noise. The goal of  $H^\infty$  estimation is to choose a causal  $F(z)$  to minimize the  $H^\infty$  norm of  $T_F(z)$ . Since the  $H^\infty$  norm of a stable LTI system is the square-root of its maximum energy gain (more precisely, its  $l^2$ -induced norm), the goal in  $H^\infty$  estimation is to solve the problem:

$$\inf_{\text{causal } F(\cdot)} \frac{\sum_i (x_{i-d} - \hat{x}_{i-d})^* (x_{i-d} - \hat{x}_{i-d})}{\sum_i x_i^* x_i + \sigma^{-2} \sum_i n_i^* n_i} \triangleq \gamma_{opt}^2. \quad (4)$$

We should further remark that the frequency domain characterization of the  $H^\infty$  norm is given by

$$\|T_F(z)\|_\infty = \sup_{0 \leq \omega \leq 2\pi} \bar{\sigma}[T_F(e^{j\omega})], \quad (5)$$

where  $\bar{\sigma}(\cdot)$  denotes maximum singular value of its argument.

There is also a related noncausal  $H^\infty$  estimation problem (which is the same estimation problem as (6) only with the causality constraint removed) which is easy to solve (see, e.g., [9]) and has the optimal norm

$$\gamma_s = \|L(z)[I + \sigma^{-2}H^*(z^{-*})H(z)]^{-1}L^*(z^{-*})\|_\infty^{1/2}, \quad (6)$$

while one corresponding noncausal  $H^\infty$ -optimal solution is given by the Wiener smoother,

$$F_s(z) = L(z)H^*(z^{-*})[\sigma^2 I + H(z)H^*(z^{-*})]^{-1}. \quad (7)$$

Note, clearly, that  $\gamma_s \leq \gamma_{opt}$  since noncausal solutions should outperform causal ones.

We notice that for the special case of a paraunitary analysis matrix  $H(z)$ , i.e., when  $H(z)H^*(z^{-*}) = H^*(z^{-*})H(z) = I$ , we have

$$\gamma_s^2 = \frac{\sigma^2}{1 + \sigma^2}$$

and

$$F_s(z) = \frac{1}{1 + \sigma^2} L(z)H^*(z^{-*})$$

Causal  $H^\infty$  estimation has been studied in [9] and conditions for the existence of solutions derived. However, explicit closed-form expressions for  $\gamma_{opt}$ , the minimum achievable disturbance attenuation, are not always available. Nonetheless, for paraunitary matrices  $H(z)$ , an expression for  $\gamma_{opt}$  can be obtained. To this end, consider the  $H^\infty$  suboptimal problem of solving for a causal estimator  $F(z)$  which achieves

$$\|[L(z) - F(z)H(z) \quad -\sigma F(z)]\|_\infty < \gamma \quad (8)$$

Assuming a paraunitary  $H(z)$ , (8) implies

$$[L(z) - F(z)H(z)][L(z) - F(z)H(z)]^* + \sigma^2 F(z)F^*(z^{-*}) < \gamma^2$$

or, equivalently,

$$[L(z) - F(z)H(z)]H^*(z^{-*})H(z)[L(z) - F(z)H(z)]^* + \sigma^2 F(z)F^*(z^{-*}) < \gamma^2,$$

and, after some algebraic simplifications,

$$\left[ \frac{L(z)H^*(z^{-*})}{\sqrt{1 + \sigma^2}} - \sqrt{1 + \sigma^2}F(z) \right] \left[ \frac{L(z)H^*(z^{-*})}{\sqrt{1 + \sigma^2}} - \sqrt{1 + \sigma^2}F(z) \right]^*$$

$$+ \frac{\sigma^2}{1+\sigma^2} < \gamma^2.$$

Introducing  $(\gamma')^2 = \gamma^2 - \frac{\sigma^2}{1+\sigma^2}$ , the last inequality implies

$$\left\| \frac{1}{\sqrt{1+\sigma^2}} L(z) H^*(z^{-*}) - \sqrt{1+\sigma^2} F(z) \right\|_{\infty} < \gamma' \quad (9)$$

Let us denote  $\frac{1}{\sqrt{1+\sigma^2}} L(z) H^*(z^{-*}) = T(z)$ . Our goal is to find a causal  $F(z)$  which minimizes (9). Define the Hankel operator associated with  $T(z) = \sum_{i=-\infty}^{\infty} t_i z^{-i}$  as

$$\mathcal{H}_T = \begin{bmatrix} t_{-1} & t_{-2} & t_{-3} & \cdots \\ t_{-2} & t_{-3} & \cdots & \\ t_{-3} & \vdots & & \\ \vdots & & & \end{bmatrix}$$

By Nehari's theorem ([10]),

$$\inf_{\text{causal } F(z)} \| [T(z) - \sqrt{1+\sigma^2} F(z)] \|_{\infty} = \bar{\sigma}(\mathcal{H}_T) \quad (10)$$

Hence the achievable  $\gamma$ -level in (5) is given by

$$\gamma_{opt}^2 = \frac{\sigma^2}{1+\sigma^2} + \bar{\sigma}^2(\mathcal{H}_T) \quad (11)$$

Clearly, when in addition  $H(z)$  is FIR, if the delay  $d$  is greater than the length of  $H(z)$ , the system  $T(z)$  is going to be causal so that the Hankel operator is zero,  $\gamma_{opt}^2 = \sigma^2/(1+\sigma^2)$ , and the optimal solution is given by:

$$F(z) = \frac{1}{1+\sigma^2} L(z) H^*(z^{-*}). \quad (12)$$

Hence, when  $H(z)$  is paraunitary and FIR, a delay equal to the length of  $H(z)$  suffices to obtain the same performance as the non-causal solution.

For IIR filters, or for delays less than the length of the FIR  $H(z)$ , (11) offers a way of relating the achievable  $\gamma$  performance of the filter bank system to the delay  $d$ . We should remark that computation of the Hankel norm of a transfer matrix is straightforward and simply requires computing the maximum singular value of a  $n \times n$  matrix, where  $n$  is the transfer matrix' McMillan degree.

It is also of interest to obtain those values of delay  $d$  that ensures that the performance of the estimator exceeds that of doing no estimation. In other words: of determining the value of delay that ensures  $\gamma_{opt} < 1$  (since  $F(z) = 0$  results in  $\gamma = 1$ ). This "worst-case non-estimability" has been studied in [11]-[12], from which it follows that choosing  $d$  greater than or equal to the number of non-minimum phase zeros of  $H(z)$  is a sufficient condition for achieving  $\gamma_{opt} < 1$ . [In all the numerical examples performed, it was observed that this condition is also necessary.]

Finally, once  $\gamma_{opt}$  has been obtained, the actual filters  $F(z)$  can be found via standard  $H^{\infty}$  techniques.

### 3. SIMULATION RESULTS

We first consider the performance of the  $H^{\infty}$  optimal synthesis filters given IIR analysis filters. The fifth order Butterworth filters shown in Figure 3 are chosen for the analysis filters (note that the  $H^{\infty}$  approach does not put any constraint on the choice of the analysis filters; the Butterworth filters are chosen for simplicity). The noise in the sub-

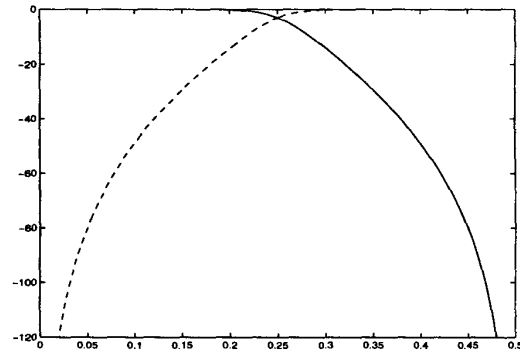


Figure 3: Frequency response of analysis filters

bands was simulated as both additive white Gaussian noise and quantization noise; in the latter case, the noise variance used for the design was approximated as  $\sigma = q^2/12$ , where  $[-q/2, q/2]$  is the range of the possible quantizer errors. The designed  $H^{\infty}$  optimal synthesis filters for  $\sigma^2 = 0.1$  are shown in Figure 4. For the performance comparison we adopt the SNR of the input signal to the reconstruction error ([7],[13])

$$SNR_r = 10 \log_{10} \left( \frac{\sum_k x^2(k)}{\sum_k (x(k-d) - \hat{x}(k))^2} \right)$$

Preliminary simulation results imply that for large  $d$ , the performance of the Kalman filter and  $H^{\infty}$  design coincide. This is expected since both filters, as  $d$  increases, converge to the Wiener smoother. As  $d$  is decreased, Kalman filter marginally outperforms  $H^{\infty}$  optimal synthesis filters. The performance comparison is shown in Table 1.

We have also performed the design of the synthesis filters given FIR analysis filters. The analysis filters are 32-tap linear phase filters adopted from [13]. Both the  $H^{\infty}$  optimal synthesis filters and Kalman filter, as well as the suggested solution from [13], perform identically when  $d$  is greater than or equal to the length of the FIR filter, i.e.,  $d \geq 32$  (of course, solution from [13] does not consider the presence of the noise, so it has to be scaled according to the

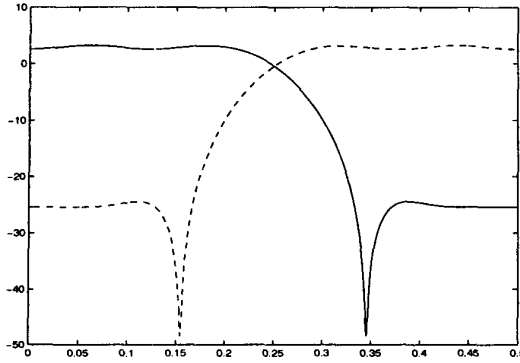


Figure 4: Frequency response of  $H^\infty$  optimal synthesis filters

d	3	7	11
Kalman $SNR_r$	13.3 dB	31.5 dB	42.2 dB
$H^\infty$ $SNR_r$	12.8 dB	31.2 dB	42.1 dB

Table 1:  $SNR_r$  comparison of the filter banks with the 3<sup>rd</sup> order Butterworth analysis filters for various  $d$

discussion in Section 2). For  $d < 32$ , the Kalman filter outperforms the  $H^\infty$  filter. For example, when the input signal with  $rms(x) = 1$  was quantized in the subbands with a precision of 8 bits, the performance comparison is shown in Table 2.

#### 4. SUMMARY

The design of multirate filter banks often assumes that the subbands of the filter bank are noise free. However, quantization and encoding cause the corruption of the signal in the subbands and may thus significantly deteriorate the reconstructed signal. Moreover, the statistical properties of such mechanisms are often hard to determine so that statistical methods (such as the Kalman filter) for reconstructing the corrupted signal are not always applicable.

d	$H^\infty$ $SNR_r$	Kalman $SNR_r$
21	23.6 dB	24.9 dB
32	42.9 dB	42.9 dB

Table 2:  $SNR_r$  comparison of the filter banks with the analysis filters as in [13] for various  $d$

In this paper, we have attempted to address the signal reconstruction problem from an  $H^\infty$  estimation point of view, which provides robustness against statistical uncertainty. We give an explicit solution for the case of orthogonal analysis filters.

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